

Optimal Sensor Configurations for Rectangular Target Detection

François-Alex Bourque and Bao U. Nguyen

Abstract—Optimal search strategies where targets are observed at several different angles are found. Targets are assumed to exhibit rectangular symmetry and have a uniformly-distributed orientation. By rectangular symmetry, it is meant that one side of a target is the mirror image of its opposite side. Finding an optimal solution is generally a hard problem. Fortunately, symmetry principles allow analytical and intuitive solutions to be found. One such optimal search strategy consists of choosing n angles evenly separated on the half-circle and leads to a lower bound of the probability of not detecting targets. As no prior knowledge of the target orientation is required, such search strategies are also robust, a desirable feature in search and detection missions.

I. INTRODUCTION

In mine hunting operations it is known that the detection performance improves when a target is observed many times at different aspect angles [1]–[6]. Similarly, classification algorithms [7]–[9] and sensor-arrays deployed for target localization and tracking [10]–[13] benefit from multi-aspect observations. This fact is, however, often overlooked. For example, the formula for the probability of detecting a target in a random search derived by Koopman is widely used yet it assumes no angular dependence [14].

In this paper, search strategies that minimize the overall probability of not detecting a target observed at several different angles are identified. Finding an optimal angular configuration is a priori intractable as it is multi-dimensional in the sense that each observation is independent of one another and hence each observation angle must be considered as a separate dimension. What is more, the explicit expression for the overall probability of detection can be hopelessly complicated even when the probability of detection for a single observation is simple and the number of observations is small.

The novelty of our approach lies in the fact that this normally intractable problem is solved using an elegant symmetry argument. Specifically, targets are assumed to exhibit rectangular symmetry. That is, the left-hand side of a target is the mirror image of its right-hand side, and its rear end is the mirror image of its front end. Many targets can be approximated with this class of symmetry including canoes, ships, submarines, mines and human bodies.

Optimal angles are then shown to be evenly distributed on multiples of the half-circle as in Ref. [13]. However, this constitutes a departure from the current literature on sensor

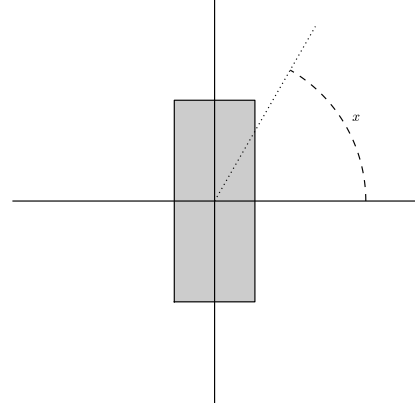


Fig. 1. Rectangular target observed at angle x .

geometry [10]–[13] as our result is derived for finite-extent targets rather than for point targets. The simplicity of the solution implies that no complicated calculations are required prior to a search as long as the target has the assumed approximate symmetry. This fact should improve the task of planning the path of mobile sensors, such as unmanned vehicles, to search for fixed targets, as well as of deploying a fixed sensor array to monitor traffic through choke points.

Assumptions and the minimization problem are stated in Section II, while Section III presents a set of search strategies that minimizes the probability of no detection. Section IV provides a lower bound of not detecting targets achievable with these strategies, which is illustrated with a specific example in Section V. Conclusions including future work are discussed in Section VI. Supplementary lemmata used in Sections III and IV are found in the Appendix.

II. PROBLEM STATEMENT

The dependence of detection process on angle occurs often in search and detection operations. In general, the effectiveness of such an operation also depends on the distance between the sensor and the target. However, here, the probability of detection is assumed constant as a function of range and, hence, the focus is only on the angular dependence. For more details on the range dependence, refer to Ref. [6].¹

F.-A. Bourque and B. U. Nguyen are scientists with Defence Research and Development Canada Centre for Operational Research and Analysis, National Defence Headquarters, 101 Colonel By drive, Ottawa, Canada.

Email enquiries should be sent to alex.bourque@drdc-rddc.gc.ca.

¹Note that the probability of detection as a function of range is primarily a characteristic of the sensor, while the probability of detection as a function of angle is primarily a characteristic of the target.

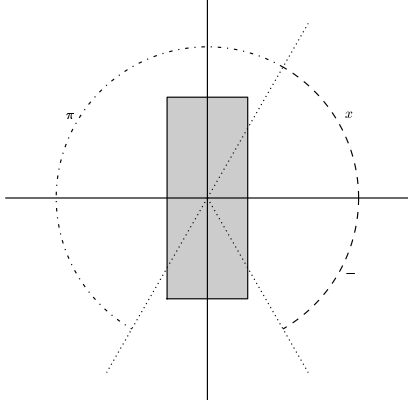


Fig. 2. Symmetries of the target: Reflection through the short axis of the target (dashed line) and reflection through the center of the target (dot-dashed line).

As shown in Fig. 1, the problem is modeled on a two-dimensional plan and the observation angle, x , is defined as the counter-clockwise angle measured in radian between the sensor beam and the short axis of a rectangular (positive horizontal axis) target. An observation angle of zero degree corresponds to the observation of the long side of the target, while an observation angle of $\pi/2$ degrees corresponds to the observation of the short side of the target. Targets considered will have approximate rectangular symmetries as shown in Fig. 2. That is, they possess a reflection axis through their short axis (left-right mirror symmetry) and a point reflection through their center.²

In what follows, the probability of no detection rather than the probability of detection is considered; one being the complement of the other. Define the single probability of no detection as the probability of not detecting the target at angle x and denote this single-value real function as $g(x)$. Note that the single probability of no detection is even due to the reflection symmetry through the short axis of the target and periodic due to the reflection through the center of the target. Specifically,

$$\begin{aligned} g(x) &= g(-x), \\ g(x) &= g(x + \pi). \end{aligned}$$

Next, define the multiple probability of no detection as the probability of not detecting a target after n observations. Let x be the orientation of the target. Let u_i be the i -th angle at which the target is observed relative to x and $\vec{\mu} = (\mu_0, \dots, \mu_{n-1})$ be the vector of the n relative observation angles. Assume the multiple observation detection process is a Bernoulli process, i.e., all observations are independent. Then, the multi-observation probability of no detection is modeled as the product of single probabilities

²Note that the composition of a reflection through the short axis followed by a reflection through the center of the target is equivalent to a reflection through the long axis of the target (forward/backward mirror symmetry).

of no detection. In general, however, the exact value of x , i.e., the orientation of the target is unknown. To circumvent this problem, assume that the target's orientation is uniformly distributed and evaluate the average multiple probability of no detection by $G(\vec{\mu})$. Then,

$$G(\vec{\mu}) = \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} dx \prod_{i=0}^{n-1} g(x + \mu_i). \quad (1)$$

Therefore, the problem amounts to finding search strategies that minimize $G(\vec{\mu})$. For simplicity, the probability of no detection is taken to mean the average multiple probability of no detection in what follows.

III. A SET OF OPTIMAL SEARCH STRATEGIES

In this section, a condition ensuring that all partial derivatives of the probability of no detection are equal to zero is derived. From this condition, a set of optimal search strategies is then identified and the probability of no detection is recast into a form used in the subsequent section.

Let us first introduce some useful notation and definitions. Let $i \in \{0, \dots, n-1\}$, $\mathcal{N}_i = \{0, \dots, n-1\} \setminus \{i\}$ and $j \in \mathcal{N}_i$. Define $\partial_i = \partial/\partial \mu_i$. Denote $\vec{\mu}^*$ as an optimal point of $G(\vec{\mu})$. Let a be an integer and b be a positive integer. Define the modulo operation as

$$a \bmod b = a - \left\lfloor \frac{a}{b} \right\rfloor b.$$

Let m be a non-negative integer and $\mu = \pi/n$. Define

$$\tilde{G}(m, n) = \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} dx \prod_{i=0}^{n-1} g(x + m\mu_i).$$

Then the following holds.

Lemma 3.1: The partial derivative can be written as

$$\begin{aligned} \partial_i G(\vec{\mu}) &= \frac{1}{2\pi} \int_{-\pi/2}^{\pi/2} dx g'(x) \left\{ \prod_{j \in \mathcal{N}_i} g(x + \mu_j - \mu_i) \right. \\ &\quad \left. - \prod_{j \in \mathcal{N}_i} g(x - \mu_j + \mu_i) \right\}. \end{aligned}$$

Proof: Apply the partial derivative to the expression for $G(\vec{\mu})$ given by (1). Then

$$\partial_i G(\vec{\mu}) = \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} dx g'(x + \mu_j) \prod_{j \in \mathcal{N}_i} g(x + \mu_j).$$

Let $x \rightarrow x - \mu_i$ and note that Lemma A.1 applies and dictates that the integral is invariant under this change of variable. Thus,

$$\partial_i G(\vec{\mu}) = \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} dx g'(x) \prod_{j \in \mathcal{N}_i} g(x + \mu_j - \mu_i). \quad (2)$$

Because $g(x)$ is even,

$$\partial_i G(\vec{\mu}) = \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} dx g'(x) \prod_{j \in \mathcal{N}_i} g(-x - \mu_j + \mu_i).$$

Let $x \rightarrow -x$ and remark that $g'(x)$ is odd. Then

$$\partial_i G(\vec{\mu}) = -\frac{1}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} dx g'(x) \prod_{j \in \mathcal{N}_i} g(x - \mu_j + \mu_i). \quad (3)$$

And the result follows from the average of (2) and (3). ■

A condition for optimizing $G(\vec{\mu})$ is thus for the integrands of all partial derivatives to be equal to zero. The next Lemma identifies a set of search strategies for which this condition holds, i.e., optimizes the probably of no detection.

Lemma 3.2: Define (m, n) to be the search strategy such that the separation between two consecutive observations is a constant and equal to $m\pi/n$ with m a positive integer. Then the search strategy (m, n) is an optimum of $G(\vec{\mu})$ and defines a subset of all possible optimal search strategies.

Proof: Evaluate the product in (2) at point $\vec{\mu}^*$ giving

$$\prod_{j \in \mathcal{N}_i} g(x + \mu_j^* - \mu_i^*).$$

Assume that (m, n) is an optimal search strategy. Then, the definition of the strategy (m, n) implies that

$$\mu_j^* - \mu_i^* = m[-(2i - j) + i]\mu.$$

From definition of the modulo note that

$$2i - j = \left\lfloor \frac{2i - j}{n} \right\rfloor n + (2i - j) \bmod n.$$

Define $\sigma_i(j) = (2i - j) \bmod n$ and recall that $g(x)$ is periodical. Then,

$$g(x + \mu_j^* - \mu_i^*) = g(x - m\sigma_i(j)\mu + mi\mu).$$

Lemma A.2 implies that the map $\sigma_i(j)$ is a bijection from the set of j to itself. Therefore,

$$\begin{aligned} \prod_{j \in \mathcal{N}_i} g(x + \mu_j^* - \mu_i^*) &= \prod_{j \in \mathcal{N}_i} g(x - m\sigma_i(j)\mu + mi\mu) \\ &= \prod_{j \in \mathcal{N}_i} g(x - mj\mu + mi\mu) \\ &= \prod_{j \in \mathcal{N}_i} g(x - \mu_j^* + \mu_i^*). \end{aligned}$$

And Lemma 3.1 then entails that $\partial_i G(\vec{\mu}^*) = 0$ for all i . ■

For an optimal search strategy (m, n) , it will be useful in what follows to cast the integral in the following form.

Lemma 3.3: Let the search strategy be (m, n) . Then the probability of no detection is equal to $\tilde{G}(m, n)$.

Proof: First, remark that only the difference between two consecutive observations is specified in Lemma 3.2. Thus, liberty exists in the choice of the absolute reference angle. Without loss of generality, take μ_0^* to be this absolute reference angle. Then,

$$\mu_i^* = \mu_0^* + mi\mu.$$

Substituting this expression into (1) yields

$$G(\vec{\mu}^*) = \frac{1}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} dx \prod_{i=0}^{n-1} g(x + \mu_0^* + mi\mu).$$

Next, let $x \rightarrow x - \mu_0^*$. Then Lemma A.1 entails that the domain of integration is invariant under such a shift of the integration variable. ■

IV. A LOWER BOUND OF THE PROBABILITY OF NO DETECTION

In the previous section, a set of optimal search strategies was identified, the (m, n) search strategies. In this section, a lower bound of the probability of no detection achievable with these strategies is proven.

Let us first introduce some useful notation and definitions. Let $\gcd(\cdot, \cdot)$ be the greatest common divider. Let r, q and p be strictly positive integers such that $m = pq$, $n = pr$ and $p = \gcd(m, n)$. Let $i \in \{0, \dots, n-1\}$, $j \in \{0, \dots, r-1\}$ and $k \in \{1, \dots, p\}$. Define

$$h(x) = g(x)g(x + m\mu) \dots g(x + (r-1)m\mu).$$

Lemma 4.1: The following identities hold:

$$\prod_{k=1}^p h(x + (k-1)\mu) = \prod_{i=0}^{n-1} g(x + i\mu), \quad (4)$$

$$\prod_{i=0}^{n-1} g(x + mi\mu) = h(x)^p. \quad (5)$$

Proof: Consider the first identity. Using the definition of $h(x)$, write the left-hand side of (4) as

$$\prod_{k=1}^p \prod_{j=0}^{r-1} g(x + mj\mu + (k-1)\mu).$$

The definitions of m and n and of the modulo operation imply that

$$mj = n \left\lfloor \frac{qj}{r} \right\rfloor + p(qj \bmod r).$$

Using this expression for mj and the periodicity of $g(x)$, the right-hand side further becomes equal to

$$\prod_{k=1}^p \prod_{j=0}^{r-1} g(x + p(qj \bmod r)\mu + (k-1)\mu).$$

Next, from Lemma A.3, the map $\sigma(j) = qj \bmod r$ is known to be a bijection from the set of j to itself. Therefore, the product can also be written as

$$\prod_{k=1}^p \prod_{j=0}^{r-1} g(x + pj\mu + (k-1)\mu).$$

Finally, from Lemma A.4, the set of (j, k) pairs can be mapped to the set of i using the bijection $\sigma(k-1, j) = (k-1) + pj$. Therefore,

$$\prod_{k=1}^p \prod_{j=0}^{r-1} g(x + pj\mu + (k-1)\mu) = \prod_{i=0}^{n-1} g(x + i\mu).$$

Now, consider the second identity. From Lemma A.4, the set of i can be mapped to the set of (j, k) pairs using the bijection $\sigma(j, k-1) = j + r(k-1)$. Therefore, the left-hand side of (5) becomes

$$\prod_{k=1}^p \prod_{j=0}^{r-1} g(x + mj\mu + m(k-1)r\mu).$$

Next, note that the definitions of m and n imply $mr = nq$ and that the definition of μ implies $n\mu = \pi$, from which follows that $m(k-1)r\mu = (k-1)q\pi$. This equality and the periodicity of $g(x)$, then imply that

$$g(x + mj\mu + m(k-1)r\mu) = g(x + mj\mu).$$

Finally, from the definition of $h(x)$,

$$\prod_{k=1}^p \prod_{j=0}^{r-1} g(x + mj\mu) = \prod_{k=1}^p h(x) = h(x)^p.$$

Theorem 4.2 (Lower Bound Estimate): For any (m, n) search strategies,

$$\tilde{G}(m, n) \geq \tilde{G}(1, n).$$

Proof: Consider a given (m, n) pair. Then, Lemma 4.1 allows to write

$$\begin{aligned} \tilde{G}(1, n) &= \frac{1}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} dx \prod_{k=1}^p h(x + (k-1)\mu), \\ \tilde{G}(m, n) &= \frac{1}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} dx h(x)^p. \end{aligned}$$

Next, let that $l \in \{0, \dots, p-1\}$ and define

$$\lambda_l(x) = \prod_{k=1}^{p-l} h(x + (k-1)\mu).$$

Assume that

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} dx \lambda_l(x) h(x)^l \leq \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} dx \lambda_{l+1}(x) h(x)^{l+1}.$$

Then

$$\begin{aligned} \tilde{G}(1, n) &= \frac{1}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} dx \lambda_0(x) \leq \frac{1}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} dx \lambda_1(x) h(x) \leq \dots \\ &\leq \frac{1}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} dx \lambda_l(x) h(x)^l \leq \dots \\ &\leq \frac{1}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} dx \lambda_{p-1}(x) h(x)^{p-1} = \frac{1}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} dx h(x)^p \\ &= \tilde{G}(m, n). \end{aligned}$$

To show that the assumption holds true, proceed by generating a partially ordered set. Let $\delta_l = (p-l-1)\mu$ and note that $\lambda_l(x) = \lambda_{l+1}(x) h(x + \delta)$. Then, the first element of the partially ordered set is

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} dx \lambda_{l+1}(x) h(x + \delta) h(x)^l.$$

And the last element is

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} dx \lambda_{l+1}(x) h(x)^{l+1}.$$

For greater clarity, the indice of $\lambda_{l+1}(x)$ and δ_l are suppressed in what follows. Next, apply recursively t times $h(x)^a h(x + \delta)^b \leq \frac{1}{2} h(x)^{a-b} [h(x)^{2a} + h(x + \delta)^{2a}]$

from the first element. After summing the resulting geometric series, the element generated is

$$\begin{aligned} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} dx \lambda(x) \left\{ \left[1 - \left(\frac{1}{2} \right)^t \right] h(x)^{l+1} \right. \\ \left. + \left(\frac{1}{2} \right)^t h(x)^{l+1-2^t} h(x + \delta)^{2^t} \right\}. \end{aligned}$$

Remark however that beyond a recursion step defined such that $2^{t+1} > l+1 \geq 2^t$, the power of $h(x)$ becomes negative in the second term. This is problematic because the last element of the partially ordered set has only positive powers of $h(x)$. To circumvent this problem, let $x \rightarrow x - \delta$ in the the second term of the element. Since the domain of integration remains unchanged (Lemma A.1), $h(x)$ is even (Lemma A.5), and $\lambda(x) = \lambda(x - \delta)$ (Corollary A.6), then the second term is also equal to

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} dx \lambda(-x) h(-x)^{l+1-2^t} h(-x + \delta)^{2^t}$$

Finally, letting $x \rightarrow -x$, the t -element becomes

$$\begin{aligned} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} dx \lambda(x) \left\{ \left[1 - \left(\frac{1}{2} \right)^t \right] h(x)^{l+1} \right. \\ \left. + \left(\frac{1}{2} \right)^t h(x + \delta)^{l+1-2^t} h(x)^{2^t} \right\} \end{aligned}$$

And the power of $h(x)$ is now greater than or equal to that of $h(x + \delta)$ in the second term allowing again the recursive application of the inequality.

The algorithm thus loops through successive recursion steps and changes of variable. Note that for $l+1 = 2^t$, the last element is generated after the first iteration. Lemma A.7 states that this is the only case for which the algorithm terminates in a finite number of iterations. For all other cases, the last element arises by letting the number of iterations go to infinity. ■

V. AN ANALYTICAL EXAMPLE

In the previous section, a lower bound of the probability of no detection was found for an arbitrary $g(x)$. In this section, the probability of no detection is evaluated and the inequality of Theorem 4.2 is explicitly verified when $g(x) = \sin(x)^2$

Lemma 5.1: Let $g(x) = \sin(x)^2$. Then

$$\tilde{G}(m, n) = \frac{2^p (2p-1)!!}{4^n p!}. \quad (6)$$

Proof: Recall that

$$\tilde{G}(m, n) = \frac{1}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} dx \prod_{k=1}^p \prod_{j=0}^{r-1} g(x + jm\mu).$$

Then Lemma A.8 implies that

$$\tilde{G}(m, n) = \frac{1}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} dx \prod_{k=1}^p \prod_{j=0}^{r-1} g\left(x + \frac{j\pi}{r}\right).$$

Let $g(x) = \sin(x)^2$ and use the identity [15]:

$$\sin(rx) = 2^{r-1} \prod_{j=0}^{r-1} \sin\left(x + \frac{j\pi}{r}\right).$$

Then,

$$\begin{aligned} \tilde{G}(m, n) &= \frac{1}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} dx \prod_{k=1}^p \frac{1}{4^{r-1}} \sin(rx)^2 \\ &= \frac{1}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} dx \frac{1}{4^{p(r-1)}} \sin(rx)^{2p} \end{aligned}$$

Finally, carrying out the integral [16] and recalling that $n = pr$,

$$\tilde{G}(m, n) = \frac{1}{4^{n-p}} \frac{(2p-1)!!}{2p!!}.$$

Since $2p!! = 2^p p!$, (6) follows. ■

Corollary 5.2: For $p = \gcd(1, n) = 1$,

$$\tilde{G}(1, n) = \frac{2}{4^n}.$$

And Theorem 4.2 follows since $(2p-1)!! = 1 \times 3 \times \dots \times (2p-1) \geq 1 \times 2 \times \dots \times p = p!$.

VI. CONCLUSIONS AND FUTURE WORK

In this paper, the angular dependence of the detection process which is often overlooked for search and detection mission is explicitly accounted for by assuming that the target possesses rectangular symmetry. One major consequence of this approximate symmetry is that the long side of a target is endowed with the largest cross section, which results in the highest probability of detection given that the target is observed only once. However, since the orientation of the target is in general unknown, there is likelihood that it will be imaged on the short side, i.e., the smallest cross-section. Therefore, the probability of not detecting the target may not be zero even if the search area is entirely covered. Making several observations of the target in order to increase the change of observing its long side is one way to address this problem.

Assuming that the observations are independent, an optimal search strategy is then to observe the target such that the separation between two consecutive observations is a constant and equal to a multiple of 180 degrees divided by the number of observations. The resulting tactic is simple, intuitive and robust (as no prior knowledge of the target orientation is required). For example, two observations separated by 90 degrees or three observations separated by 60 degrees will minimize the probability of no detection.

Having shown that one of these search strategies leads to a lower bound of the probability of no detection, work is currently underway to prove that it is also a globally minimal search strategy. Another logical extension of this work is to relax the assumption that information provided by subsequent observations is uncorrelated (i.e., follows a Bernoulli process) as it entails that the probability of not detecting a target decreases with increasing number of observations even if the observations are co-linear. This is questionable as no additional information is gained.

VII. ACKNOWLEDGMENTS

One of the authors (A. Bourque) would like to acknowledge A. Percival from Defence R&D Canada - Atlantic for bringing to his attention the issue of correlation effects in the detection process.

APPENDIX SUPPLEMENTARY LEMMATA

Lemma A.1 (Rotational Invariance): Let $\omega \in \mathbb{R}$ and $h(x) = h(x + \pi)$. Then

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} dx h(x - \omega) = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} dx h(x).$$

Proof: Let $x \rightarrow x + \omega$ on the RHS and break the integration interval of the resulting integral into $[-\frac{\pi}{2} - \omega, -\frac{\pi}{2}]$ and $[-\frac{\pi}{2}, \frac{\pi}{2} - \omega]$. Then let $x \rightarrow x - \pi$ in the integral over the first interval and recall that by assumption $h(x)$ is periodic. ■

Lemma A.2: Let $i \in \{0, \dots, n-1\}$, $\mathcal{N}_i = \{0, \dots, n-1\} \setminus \{i\}$ and $j \in \mathcal{N}_i$. Then $\sigma_i(j) = (2i - j) \bmod n$ is a bijection from \mathcal{N}_i to itself and its own inverse.

Proof: Composition gives

$$\sigma_i(\sigma_i(j)) = (2i - (2i - j) \bmod n) \bmod n.$$

Use the definition of the modulo twice to give

$$\sigma_i(\sigma_i(j)) = j + \left\lfloor \frac{(2i-j)}{n} \right\rfloor n - \left\lfloor \frac{j + \left\lfloor \frac{(2i-j)}{n} \right\rfloor n}{n} \right\rfloor n.$$

Recall that $j \in \mathcal{N}_i$ and note that $\frac{j}{n} < 1$. Then $\left\lfloor \frac{j}{n} + \left\lfloor \frac{(2i-j)}{n} \right\rfloor \right\rfloor n = \left\lfloor \frac{(2i-j)}{n} \right\rfloor n$ and $\sigma_i(\sigma_i(j)) = j$. ■

Lemma A.3: Let r, q be positive integers such that $\gcd(r, q) = 1$, i.e., r and q are co-primes. Let $i \in \{0, \dots, r-1\}$. Then the map $\sigma(i) = qi \bmod r$ is a bijection of the set of i to itself.

Proof: Proceed with a proof by contradiction. Assume this map is not a bijection. Then there exists a pair $u, v \in \{0, \dots, r-1\}$ such that $u \neq v$ and $qu \bmod r = qv \bmod r$. Next, assume that $u > v$ then $q(u-v) \bmod r = qw \bmod r = 0$ where $w = u-v$. This implies that $qw = ry$ with y a positive integer, as $w, q > 0$. Because q and r are co-primes, i.e., $\gcd(r, q) = 1$ then $w = rz$ with $z > 0$, which leads to a contraction as $w = u-v < r-1$. ■

Lemma A.4: Let $u \in \{0, \dots, a-1\}$, $v \in \{0, \dots, b-1\}$, and $w \in \{0, \dots, n-1\}$ where $n = ab$. Then $\sigma(u, v) = u + av$ and $\sigma^{-1}(w) = (w \bmod a, \left\lfloor \frac{w}{a} \right\rfloor)$ are bijections.

Proof: Composition gives

$$\sigma(\sigma^{-1}(w)) = w \bmod a + a \left\lfloor \frac{w}{a} \right\rfloor = w,$$

where the last equality follows from the definition of the modulo operation. Similarly,

$$\sigma^{-1}(\sigma(u, v)) = \left[(u + av) \bmod a, \left\lfloor \frac{u + av}{a} \right\rfloor \right] = (u, v),$$

where the last equality follows from the definition of the modulo operation and since $u < a$. Therefore, $\sigma(u, v)$ and

$\sigma^{-1}(w)$ are both one-to-one, onto, and inverse of each other. ■

Lemma A.5: $h(x) = h(-x)$.

Proof: Because of the periodicity of $g(x)$ and $nq = mr$, $g(x + mj\mu) = g(x + m(j-r)\mu)$. Let $j \rightarrow -j+r$. Then $h(x) = g(x) \dots g(x - (r-1)j\mu)$ and the proof follows since by definition $g(x)$ is even. ■

Corollary A.6: Consider $\lambda_l(x)$ and $\delta_l = (p - k - 1)\mu$. Then $\lambda_l(x - \delta_l) = h(x - (p - l - 1)\mu) \dots h(x)$ and $\lambda_l(x - \delta_l) = \lambda_l(-x)$ since $h(x)$ is even by Lemma A.5.

Lemma A.7: Let i and c be positive integers. Let $\{t_i\}$ be the set of non-negative integers such that $2^{t_i+1} > c \geq 2^{t_i}$. And let $\{u_i\}$ be the set of non-negative integer such that $u_i = c - 2^{t_i}u_{i-1}$ and $u_0 = 1$. Then, a fix point exists if and only if $c = 2^{t_1}$.

Proof: After i iterations,

$$\begin{aligned} u_i &= c - 2^{t_i}u_{i-1} = c - 2^{t_i}(c - 2^{t_{i-1}}u_{i-2}) \\ &= (c - 2^{t_i}(\dots(c - 2^{t_1})\dots)) \\ &= c[1 - 2^{t_i}(\dots(1 - 2^{t_2})\dots)] + (-1)^i 2^{t_i + \dots + t_1}. \end{aligned}$$

Then $u_i = 0$ implies that the prime factorization of $(k+1)$ must be 2^a where a is a non-negative integer. Because $2^{t_1+1} > (l+1) = 2^a \geq 2^{t_1}$, then $a = t_1$ and $u_i = 0$ for $i > 0$. ■

Lemma A.8:

$$\prod_{j=0}^{r-1} g\left(x + \frac{mj\pi}{n}\right) = \prod_{j=0}^{r-1} g\left(x + \frac{j\pi}{r}\right).$$

Proof: Recall that r , q and p are positive integers such that $m = pq$, $n = pr$, and $p = \gcd(m, n)$. Then

$$g\left(x + \frac{mj\pi}{n}\right) = g\left(x + \frac{qj\pi}{r}\right).$$

Use the periodicity of $g(x)$ and the definition of the modulo to further write the right-hand side of the equality as

$$g\left(x + \frac{qj\pi}{r} - \left\lfloor \frac{qj}{r} \right\rfloor \pi\right) = g\left(x + (qj \bmod r) \frac{\pi}{r}\right).$$

Lemma A.3 implies that map $\sigma(j) = qj \bmod r$ is a bijection from the set of j to itself. ■

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